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Approximation of a Convex Function by Convex Algebraic Polynomials in L_p, $1 \leqslant p < \infty$

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In this paper we show that the best approximation of a convex function by convex algebraic polynomials in L_p, $1 \leqslant p < \infty$, is $O(n^{-2/p})$.

1. Known Results and Main Theorem

Let us denote by H_n the set of all algebraic polynomials of degree $\leqslant n$ and by

$$E_n(f)_p = \inf\{\|f - P\|_p; P \in H_n\}$$

the best L_p approximation of f by polynomials from H_n.

We denote by $K[-1, 1]$ the set of all convex functions on $[-1, 1]$ such that $\max_{x \in [-1, 1]} f(x) = 1$, $\min_{x \in [-1, 1]} f(x) = 0$.

In [2] it is shown that the best algebraic approximation of a convex function in L_p for $1 < p < \infty$ is

$$E_n(f)_p = o(n^{-2/p}).$$

In [1] it is shown that

$$E_n(f)_1 = O(n^{-2}).$$

In these papers the type of the approximating polynomial is not discussed.

Main Theorem. *Let $f \in K[-1, 1]$. Then for every n there exists a convex polynomial Q_n of degree $\leqslant n$ such that*

$$\|f - Q_n\|_p \leqslant An^{-2/p}, \qquad 1 \leqslant p < \infty,$$

where A is a constant independent of f and p.

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2. Notation and Lemmas

We shall use the function

$$g_a(x)=\begin{cases}0, & \text{for } \ -1\leqslant x\leqslant a\\ (x-a)/(1-a), & \text{for } \ a\leqslant x\leqslant 1\end{cases}$$

and Chebyshev polynomials

$$T_n(x)=\cos(n\arccos x),\qquad x\in[-1,1].$$

We denote by $f[x_{i-1};x_i;x_{i+1}]$ the second divided difference, i.e.,

$$f[x_{i-1};x_i;x_{i+1}]=\frac{f(x_{i-1})}{(x_{i-1}-x_i)(x_{i-1}-x_{i+1})}-\frac{f(x_i)}{(x_{i-1}-x_i)(x_i-x_{i+1})}$$
$$+\frac{f(x_{i+1})}{(x_i-x_{i+1})(x_{i-1}-x_{i+1})}.$$

Lemma 1. *Let $G\geqslant 0$ on $[-1,1]$,*

$$G_1(x)=\int_{-1}^{x} G(t)\,dt,$$

$$G_2(x)=\int_{-1}^{x} G_1(t)\,dt=\int_{-1}^{x}(x-t)\,G(t)\,dt,$$

$$G_2(1)=\int_{-1}^{1}(1-t)\,G(t)\,dt=1,$$

and let the constant a be defined by

$$\int_{-1}^{1} xG(x)\,dx=a\int_{-1}^{1} G(x)\,dx. \tag{1}$$

Then

1. *G_2 is a convex function,*

$$G_2(-1)=G_2'(-1)=0,$$

$$G_2'(1)=\frac{1}{1-a},\qquad |a|<1; \tag{2}$$

2. $G_2(x)\geqslant g_a(x),\qquad x\in[-1,1]$;
3. $\int_{-1}^{1}(G_2(x)-g_a(x))\,dx=(1/2)\int_{-1}^{1}(x-a)^2\,G(x)\,dx$;
4. $|a-\xi|\leqslant\int_{-1}^{1}|x-\xi|\,G(x)\,dx/\int_{-1}^{1}G(x)\,dx$ *for arbitrary* ξ;
5. $\int_{-1}^{1}(x-a)^2\,G(x)\,dx\leqslant\int_{-1}^{1}(x-\xi)^2\,G(x)\,dx$ *for arbitrary* ξ.

Proof. 1. The convexity of G_2 follows by the construction. It is obvious that $G_2(-1)=G_2'(-1)=0$, $|a|<1$, and $G_2'(1)=1/(1-a)$, since

$$\begin{aligned}G_2'(1)&=\int_{-1}^{1} G(x)\,dx=\frac{1}{a}\int_{-1}^{1} xG(x)\,dx\\&=\frac{1}{a}\left(\int_{-1}^{1}(x-1)\,G(x)\,dx+\int_{-1}^{1} G(x)\,dx\right)\\&=\frac{1}{a}\,(-1+G_2'(1)).\end{aligned}$$

2. $G_2(x)\geqslant g_a(x)$, $x\in[-1,1]$, because $G_2\geqslant 0$ on $[-1,1]$, $G_2(1)=1$, $G_2(-1)=0$, $G_2'(1)=1/(1-a)>0$, G_2 is convex, and $g_\alpha'(x)=1/(1-a)$, $x\in(a,1]$.

3. Consider

$$\begin{aligned}I&=2\int_{-1}^{1}(G_2(x)-g_a(x))\,dx=2\int_{-1}^{1}\left(\int_{-1}^{x}(x-u)\,G(u)\,du-g_a(x)\right)dx\\&=2\left(\int_{-1}^{1}\int_{a}^{1}(x-u)\,G(u)\,dx\,du-\int_{a}^{1}(x-a)/(1-a)\,dx\right)\\&=\int_{-1}^{1}(1-u)^2\,G(u)\,du-1+a\\&=\int_{-1}^{1}((a-u)^2+2(a-u)(1-a)+(1-a)^2)\,G(u)\,du-1+a\\&=\int_{-1}^{1}(a-u)^2\,G(u)\,du+(1-a)\left(\int_{-1}^{1}(a+1-2u)\,G(u)\,du-1\right)\\&=\int_{-1}^{1}(a-u)^2\,G(u)\,du\\&\quad+(1-a)\left(2\int_{-1}^{1}(1-u)\,G(u)\,du-(1-a)\int_{-1}^{1}G(u)\,du-1\right)\\&=\int_{-1}^{1}(a-u)^2\,G(u)\,du,\end{aligned}$$

which proves part 3.

4. Let ξ be arbitrary. Then

$$|a-\xi|=\left|\frac{\int_{-1}^{1}xG(x)\,dx}{\int_{-1}^{1}G(x)\,dx}-\xi\right|=\left|\frac{\int_{-1}^{1}(x-\xi)\,G(x)\,dx}{\int_{-1}^{1}G(x)\,dx}\right|\leqslant\frac{\int_{-1}^{1}|x-\xi|\,G(x)\,dx}{\int_{-1}^{1}G(x)\,dx}.$$

Part 5 follows from

$$\begin{aligned}\int_{-1}^{1} (x-a)^2 G(x)\,dx &= \int_{-1}^{1} ((x-\xi)^2 + 2(x-\xi)(\xi-a) + (\xi-a)^2)\, G(x)\,dx \\ &= \int_{-1}^{1} (x-\xi)^2 G(x)\,dx + 2(\xi-a)\int_{-1}^{1} xG(x)\,dx \\ &\quad - 2(\xi-a)\xi \int_{-1}^{1} G(x)\,dx + (\xi-a)^2 \int_{-1}^{1} G(x)\,dx \\ &= \int_{-1}^{1} (x-\xi)^2 G(x)\,dx - (a-\xi)^2 \int_{-1}^{1} G(x)\,dx \\ &\leqslant \int_{-1}^{1} (x-\xi)^2 G(x)\,dx.\end{aligned}$$

The lemma is proved.

Let us consider the polynomials

$$R_{k,n}(x) = \left(\frac{T_n(x)}{x - x_{k,n}}\right)^{2q},$$

where

$$T_n(x) = \cos(n \arccos x), \qquad -1 \leqslant x \leqslant 1$$

and

$$x_{k,n} = \cos t_{k,n}, \qquad t_{k,n} = \frac{2k-1}{2n}\pi, \qquad k = 1, 2, ..., n;$$

$$R_{k,n} \in H_{2q(n-1)}.$$

Let $x = \cos y$, $y \in [0, \pi]$.

LEMMA 2. *For any* $x \in [-1, 1]$

$$\left|\frac{T_n(x)}{x - x_{k,n}}\right| \leqslant \pi \frac{\min\{1, n\,|y - t_{k,n}|\}}{|y - t_{k,n}|\,\sqrt{1 - x_{k,n}^2}}.$$

Proof. It is obvious that $|T_n(x)| \leqslant 1$. On the other hand

$$\begin{aligned}|T_n(x)| &= |T_n(x) - T_n(x_{k,n})| = |\cos ny - \cos nt_{k,n}| \\ &= n\,|y - t_{k,n}|\,|-\sin n\xi| \leqslant n\,|y - t_{k,n}|,\end{aligned}$$

where $\xi \in \operatorname{int}(y, t_{k,n})$. It follows that

$$|T_n(x)| \leqslant \min\{1, n\,|\,y - t_{k,n}|\}.$$

We now estimate $|x - x_{k,n}|$:

$$|x - x_{k,n}| = |\cos y - \cos t_{k,n}| = \left|2 \sin \frac{y + t_{k,n}}{2} \sin \frac{y - t_{k,n}}{2}\right|$$

$$\geqslant \frac{|y - t_{k,n}|}{\pi} \sin t_{k,n} = \frac{|y - t_{k,n}|\sqrt{1 - x_{k,n}^2}}{\pi}.$$

The fact that $\sin((y + t_{k,n})/2) \geqslant \sin t_{k,n}/2$ follows from the convexity of $\sin x$ on $[0, \pi]$.

The lemma is proved.

LEMMA 3. *There exists an absolute constant C, such that*

$$\left|\frac{T_n(x)}{x - x_{k,n}}\right| \geqslant C \frac{n}{\sqrt{1 - x_{k,n}^2}}$$

for $|y - t_{k,n}| \leqslant 1/n$.

Proof.

$$\left|\frac{T_n(x)}{x - x_{k,n}}\right| = \left|\frac{\cos ny - \cos nt_{k,n}}{\cos y - \cos t_{k,n}}\right| = \left|\frac{2 \sin(n(y + t_{k,n})/2) \sin(n(y - t_{k,n})/2)}{2 \sin((y + t_{k,n})/2) \sin((y - t_{k,n})/2)}\right|$$

$$\geqslant \left|\frac{(n(y - t_{k,n})/\pi) \sin(n(y + t_{k,n})/2)}{((y - t_{k,n})/2 \sin((y + t_{k,n})/2)}\right| \geqslant \frac{2n\,|\sin(n(y + t_{k,n})/2)|}{\pi \sin(y + t_{k,n})/2)}.$$

But $|y - t_{k,n}| \leqslant 1/n$, so $y = t_{k,n} + \varepsilon$ where $|\varepsilon| \leqslant 1/n$, $k = 1, 2, ..., n$. We have

$$\left|\sin \frac{n(y + t_{k,n})}{2}\right| = \left|\sin \frac{n(2t_{k,n} + \varepsilon)}{2}\right| = \left|\sin\left(n \frac{2k - 1}{2n} \pi + \frac{n\varepsilon}{2}\right)\right|$$

$$= \left|(-1)^k \sin\left(-\frac{\pi}{2} + \frac{n\varepsilon}{2}\right)\right| = \left|\cos \frac{n\varepsilon}{2}\right| \geqslant \cos \frac{1}{2}.$$

On the other hand

$$\sin \frac{y + t_{k,n}}{2} = \sin\left(t_{k,n} + \frac{\varepsilon}{2}\right) = \sin t_{k,n} \cos \frac{\varepsilon}{2} + \sin \frac{\varepsilon}{2} \cos t_{k,n}$$

$$\leqslant \sin t_{k,n} + \sin \frac{\varepsilon}{2} \leqslant 2 \sin t_{k,n} = 2\sqrt{1 - x_{k,n}^2}. \tag{3}$$

The lemma follows.

LEMMA 4. *For any integer q there exists a constant $C = C(q)$, such that for $|y - t_{k,n}| \leqslant 1/n$,*

$$\int_{-1}^{1} \left(\frac{T_n(x)}{x - x_{k,n}}\right)^{2q} dx \geqslant C\left(\frac{n}{\sqrt{1 - x_{k,n}^2}}\right)^{2q-1}.$$

Proof. Using Lemma 3 we obtain

$$\int_{-1}^{1} \left(\frac{T_n(x)}{x - x_{k,n}}\right)^{2q} dx = \int_0^{\pi} \left(\frac{\cos ny}{\cos y - \cos t_{k,n}}\right)^{2q} \sin y\, dy$$

$$\geqslant \int_{t_k - 1/n}^{t_k + 1/n} \left(\frac{\cos ny}{\cos y - \cos t_{k,n}}\right)^{2q} \sin y\, dy$$

$$\geqslant C\left(\frac{n}{\sqrt{1 - x_{k,n}^2}}\right)^{2q} \frac{\sin t_{k,n}}{n} = C\left(\frac{n}{\sqrt{1 - x_{k,n}^2}}\right)^{2q-1}.$$

LEMMA 5. *For any integer q, $q \geqslant 3$, there exists a constant $C = C(q)$, such that*

$$\int_{-1}^{1} |x - x_{k,n}| \left(\frac{T_n(x)}{x - x_{k,n}}\right)^{2q} dx \leqslant C\left(\frac{n}{\sqrt{1 - x_{k,n}^2}}\right)^{2q-2}.$$

Proof. Let $q \geqslant 3$, integer. Then using Lemma 2 we obtain

$$I = \int_{-1}^{1} |x - x_{k,n}| \left(\frac{T_n(x)}{x - x_{k,n}}\right)^{2q}$$

$$\leqslant \pi^{2q} \int_0^{\pi} \sin y\, |\cos y - \cos t_{k,n}| \left(\frac{\min\{1, n\,|y - t_{k,n}|\}}{|y - t_{k,n}| \sin t_{k,n}}\right)^{2q} dy$$

$$= \pi^{2q} \int_0^{\pi} \sin y\, |\cos y - \cos t_{k,n}| \left(\frac{n}{\sin t_{k,n}}\right)^{2q} \left(\frac{\min\{n^{-1}, |y - t_{k,n}|\}}{|y - t_{k,n}|}\right)^{2q} dy.$$

We split this integral into three parts

$$\int_0^{t_{k,n} - 1/n} + \int_{t_{k,n} - 1/n}^{t_{k,n} + 1/n} + \int_{t_{k,n} + 1/n}^{\pi} = I_1 + I_2 + I_3$$

and estimate each one. Let us begin with I_2:

$$I_2 = \int_{t_{k,n} - 1/n}^{t_{k,n} + 1/n} \sin y\, |\cos y - \cos t_{k,n}| \left(\frac{n}{\sin t_{k,n}}\right)^{2q} \left(\frac{\min\{n^{-1}, |y - t_{k,n}|\}}{|y - t_{k,n}|}\right)^{2q} dy$$

$$= \int_{t_{k,n} - 1/n}^{t_{k,n} + 1/n} \sin y\, |\cos y - \cos t_{k,n}| \left(\frac{n}{\sin t_{k,n}}\right)^{2q} dy$$

$$= \left(\frac{n}{\sin t_{k,n}}\right)^{2q} \int_{t_{k,n} - 1/n}^{t_{k,n} + 1/n} 2 \sin y \sin \frac{y + t_{k,n}}{2} \sin \frac{|y - t_{k,n}|}{2}\, dy$$

$$\leqslant C\left(\frac{n}{\sin t_{k,n}}\right)^{2q-2} = C\left(\frac{n}{\sqrt{1 - x_{k,n}^2}}\right)^{2q-2}.$$

Here we use that $|y-t_{k,n}| \leqslant n^{-1}$, (3), and

$$\begin{aligned}\sin y &= \sin(y-t_{k,n}+t_{k,n})\\ &= \sin(y-t_{k,n})\cos t_{k,n}+\cos(y-t_{k,n})\sin t_{k,n}\\ &\leqslant \sin(1/n)+\sin t_{k,n} \leqslant 2\sin t_{k,n}. \qquad (4)\end{aligned}$$

We now estimate I_1:

$$\begin{aligned} I_1 &= \int_0^{t_{k,n}-1/n} \sin y\,|\cos y-\cos t_{k,n}| \left(\frac{n}{\sin t_{k,n}}\right)^{2q} \left(\frac{\min\{n^{-1}, |y-t_{k,n}|\}}{|y-t_{k,n}|}\right)^{2q} dy\\ &= (\sin t_{k,n})^{-2q} \int_0^{t_{k,n}-1/n} 2(y-t_{k,n})^{-2q} \sin y \sin\frac{y+t_{k,n}}{2} \sin\frac{|y-t_{k,n}|}{2}\, dy\\ &\leqslant (\sin t_{k,n})^{-2q} \int_0^{t_{k,n}-1/n} |y-t_{k,n}|^{-2q+1} \sin y \sin\frac{y+t_{k,n}}{2}\, dy. \end{aligned}$$

Using that

$$\begin{aligned}\sin y &= \sin(y-t_{k,n})\cos t_{k,n}+\cos(y-t_{k,n})\sin t_{k,n}\\ &\leqslant |y-t_{k,n}|+\sin t_{k,n} \qquad (5)\end{aligned}$$

and

$$\begin{aligned}\sin\frac{y+t_{k,n}}{2} &= \sin t_{k,n}\cos\frac{y-t_{k,n}}{2}+\cos t_{k,n}\sin\frac{y-t_{k,n}}{2}\\ &\leqslant \sin t_{k,n}+|y-t_{k,n}| \qquad (6)\end{aligned}$$

we find

$$\begin{aligned}\sin y\sin\frac{y+t_{k,n}}{2} &\leqslant (|y-t_{k,n}|+\sin t_{k,n})^2\\ &\leqslant 2(|y-t_{k,n}|^2+(\sin t_{k,n})^2).\end{aligned}$$

Let $v=t_{k,n}-y$. Then

$$\begin{aligned} I_1 &\leqslant \frac{2}{(\sin t_{k,n})^{2q}} \int_{1/n}^{t_{k,n}} \frac{dv}{v^{2q-3}} + \frac{2}{(\sin t_{k,n})^{2q-2}} \int_{1/n}^{t_{k,n}} \frac{dv}{v^{2q-1}}\\ &\leqslant \frac{2}{(\sin t_{k,n})^{2q}} \int_{1/n}^{\infty} \frac{dv}{v^{2q-3}} + \frac{2}{(\sin t_{k,n})^{2q-2}} \int_{1/n}^{\infty} \frac{dv}{v^{2q-1}}\\ &\leqslant C\left(\frac{n}{\sin t_{k,n}}\right)^{2q-2} = C\left(\frac{n}{\sqrt{1-x_{k,n}^2}}\right)^{2q-2}. \end{aligned}$$

I_3 is estimated similarly. This establishes the lemma.

LEMMA 6. *For any integer q, $q \geqslant 4$, there exists a constant $C = C(q)$, such that*

$$\int_{-1}^{1} (x - x_{k,n})^2 \left(\frac{T_n(x)}{x - x_{k,n}}\right)^{2q} dx \leqslant C \left(\frac{n}{\sin t_{k,n}}\right)^{2q-3}.$$

Proof.

$$\begin{aligned} I &= \int_{-1}^{1} (x - x_{k,n})^2 \left(\frac{T_n(x)}{x - x_{k,n}}\right)^{2q} dx \\ &\leqslant \pi^{2q} \int_0^{\pi} \sin y (\cos y - \cos t_{k,n})^2 \left(\frac{\min\{1, n\,|y - t_{k,n}|\}}{|y - t_{k,n}| \sin t_{k,n}}\right)^{2q} dy \\ &= \pi^{2q} \left(\frac{n}{\sin t_{k,n}}\right)^{2q} \int_0^{\pi} \sin y (\cos y - \cos t_{k,n})^2 \left(\frac{\min\{n^{-1}, |y - t_{k,n}|\}}{|y - t_{k,n}|}\right)^{2q} dy. \end{aligned}$$

We split this integral into three parts

$$\int_0^{t_{k,n} - 1/n} + \int_{t_{k,n_n} + 1/n}^{t_{k,n} + 1/n} + \int_{t_{k,n} + 1/n}^{\pi} = I_1 + I_2 + I_3$$

and estimate each one. Consider first I_2. Using (3) and (4) we obtain

$$\begin{aligned} I_2 &= \pi^{2q} \left(\frac{n}{\sin t_{k,n}}\right)^{2q} \int_{t_{k,n} - 1/2}^{t_{k,n} + 1/n} \sin y (\cos y - \cos t_{k,n})^2 \, dy \\ &\leqslant C \left(\frac{n}{\sin t_{k,n}}\right)^{2q-3}. \end{aligned}$$

Now estimate I_1:

$$\begin{aligned} I_1 &= \pi^{2q} \left(\frac{n}{\sin t_{k,n}}\right)^{2q} \int_0^{t_{k,n} - 1/n} \sin y (\cos y - \cos t_{k,n})^2 \frac{dy}{(n(y - t_{k,n}))^{2q}} \\ &\leqslant \frac{\pi^{2q}}{(\sin t_{k,n})^{2q}} \int_0^{t_{k,n} - 1/n} \sin y \left(2 \sin \frac{y + t_{k,n}}{2} \sin \frac{|y - t_{k,n}|}{2}\right)^2 \frac{dy}{(y - t_{k,n})^{2q}} \\ &\leqslant \frac{\pi^{2q}}{(\sin t_{k,n})^{2q}} \int_0^{t_{k,n} - 1/n} \sin y \left(\sin \frac{y + t_{k,n}}{2}\right)^2 \frac{dy}{(y - t_{k,n})^{2q-2}}. \end{aligned}$$

Using (5), (6), and Hölder's inequality we obtain

$$\begin{aligned} \sin y \left(\sin \frac{y + t_{k,n}}{2}\right)^2 &\leqslant (|y - t_{k,n}| + \sin t_{k,n})^3 \\ &\leqslant 4(|y - t_{k,n}|^3 + (\sin t_{k,n})^3). \end{aligned}$$

So

$$I_1 \leqslant \frac{4\pi^{2q}}{(\sin t_{k,n})^{2q}} \int_0^{t_{k,n}-1/n} \frac{dy}{|y-t_{k,n}|^{2q-5}} + \frac{4\pi^{2q}}{(\sin t_{k,n})^{2q-3}} \int_0^{t_{k,n}-1/n} \frac{dy}{|y-t_{k,n}|^{2q-2}}.$$

Let $t_{k,n} - y = v$. Then

$$\begin{aligned} I_1 &\leqslant \frac{4\pi^{2q}}{(\sin t_{k,n})^{2q}} \int_{1/n}^{t_{k,n}} \frac{dv}{v^{2q-5}} + \frac{4\pi^{2q}}{(\sin t_{k,n})^{2q-3}} \int_{1/n}^{t_{k,n}} \frac{dv}{v^{2q-2}} \\ &\leqslant \frac{4\pi^{2q}}{(\sin t_{k,n})^{2q}} \int_{1/n}^{\infty} \frac{dv}{v^{2q-5}} + \frac{4\pi^{2q}}{(\sin t_{k,n})^{2q-3}} \int_{1/n}^{\infty} \frac{dv}{v^{2q-2}} \\ &\leqslant C\left(\frac{n}{\sin t_{k,n}}\right)^{2q-3}. \end{aligned}$$

I_3 is estimated in the same way. This proves the lemma.

3. THEOREMS

THEOREM 1. *There exist absolute constants C_1 and C_2, such that for any $k = r, r+1, ..., n-r$, $r = [2C_1 + 2]$, there is $a_k \in [-1, 1]$ and a convex polynomial $G_{2,k}$ of degree $\leqslant 8n-6$, $n > 2r$ such that*

(a) $|x_{k,n} - a_k| \leqslant C_1 (\sin t_{k,n})/n$;

(b) $0 \leqslant G_{2,k}(x) - g_{a_k}(x) \leqslant 1$;

(c) $\|G_{2,k} - g_{a_k}\|_p \leqslant C_2 n^{-2/p}$, $1 \leqslant p < \infty$.

Proof. Let us consider the polynomial

$$G_k(x) = \gamma_k \left(\frac{T_n(x)}{x - x_{k,n}}\right)^8,$$

$x \in [-1, 1]$, where γ_k is a constant. We want to apply Lemma 1. Denote

$$G_{2,k}(x) = \int_{-1}^{x} (x-u)\, G_k(u)\, du.$$

We choose γ_k so that

$$G_{2,k}(1) = \gamma_k \int_{-1}^{1} (1-x) \left(\frac{T_n(x)}{x - x_{k,n}}\right)^8 dx = 1,$$

i.e.,

$$\gamma_k = 1 \Big/ \int_{-1}^{1} (1-x)\left(\frac{T_n(x)}{x-x_{k,n}}\right)^8 dx$$

and

$$G'_{2,k}(1) = 1/(1-a_k),$$

where a_k is defined by (1). But

$$G'_{2,k}(1) = \int_{-1}^{1} G_k(x)\, dx = \gamma_k \int_{-1}^{1} \left(\frac{T_n(x)}{x-x_{k,n}}\right)^8 dx.$$

It follows that

$$1/(1-a_k) = \gamma_k \int_{-1}^{1} \left(\frac{T_n(x)}{x-x_{k,n}}\right)^8 dx. \tag{7}$$

We now prove (a). Let $\xi = x_{k,n}$ in Lemma 1 in part 4. Then using Lemma 4 and Lemma 5 with $q=4$ we find

$$\begin{aligned} |a_k - x_{k,n}| &\leqslant \frac{\int_{-1}^{1} |x-x_{k,n}|\, (T_n(x)/(x-x_{k,n}))^8\, dx}{\int_{-1}^{1} (T_n(x)/(x-x_{k,n}))^8\, dx} \\ &\leqslant C_1 \left(\frac{n}{\sin t_{k,n}}\right)^6 \left(\frac{\sin t_{k,n}}{n}\right)^7 = C_1 \frac{\sin t_{k,n}}{n}. \end{aligned} \tag{8}$$

The claim (b) follows by the construction and Lemma 1.

Now we estimate γ_k using (7). We have, by (8),

$$\begin{aligned} 1-|a_k| = 1-|x_{k,n}| + |x_{k,n}| - |a_k| &\geqslant \frac{1-x_{k,n}^2}{2} - |x_{k,n} - a_k| \\ &\geqslant \frac{1}{2}\sin^2 t_{k,n} - \frac{C_1}{n}\sin t_{k,n} \\ &\geqslant \frac{1}{4}\sin^2 t_{k,n}, \end{aligned} \tag{9}$$

when

$$\sin t_{k,n} \geqslant \frac{4C_1}{n},$$

which is the case when

$$\min\{k, n-k\} > 2C_1 + 1/2,$$

by the inequality $\sin t \geqslant \min\{t, \pi - t\}$, valid for $0 \leqslant t \leqslant \pi$. Let $r = [2C_1 + 2]$ and $k = r, r+1, ..., n-r$, $n > 2r$. By (7), (9), and Lemma 4 with $q = 4$ we find

$$\gamma_k \leqslant \frac{C_3}{(\sin t_{k,n})^2} \left(\frac{\sin t_{k,n}}{n}\right)^7 = C_3 \frac{(\sin t_{k,n})^5}{n^7}, \tag{10}$$

where C_3 is an absolute constant. Now estimate the approximation in L_1 and L_p. Using Lemma 1 and Lemma 6 with $q = 4$ we find

$$\begin{aligned} \|G_{2,k} - g_{a_k}\|_1 &= \int_{-1}^{1} (G_{2,k}(x) - g_{a_k}(x))\, dx \\ &\leqslant \int_{-1}^{1} (x - x_{k,n})^2\, G_2(x)\, dx \leqslant C_2 n^{-2}. \end{aligned}$$

The estimate (c) follows trivially from $\|\cdot\|_p \leqslant \|\cdot\|_\infty^{1-1/p} \|\cdot\|_1^{1/p}$. Theorem 1 is established.

Let us consider the points $x_{k,n}$, $k = r, r+1, ..., n-r$, from Theorem 1. Let C_1 be the constant from (a), Theorem 1, and $r = [2C_1 + 2]$, and assume that $n > 2r + C_1 + 1$. Choose $l < k$ so that $k - l = [C_1] + 1$ and $r \leqslant l < k \leqslant n - r$. Then $x_{k,n} < x_{l,n}$, and

$$x_{l,n} - x_{k,n} \geqslant C_1 \frac{\sin t_{k,n}}{n},$$

since

$$\begin{aligned} x_{l,n} - x_{k,n} &= \cos t_{l,n} - \cos t_{k,n} \\ &= 2 \sin \frac{t_{k,n} + t_{l,n}}{2} \sin \frac{t_{k,n} - t_{l,n}}{2} \\ &\geqslant 2\, \frac{t_{k,n} - t_{l,n}}{\pi} \sin \frac{t_{k,n} + t_{l,n}}{2} \\ &= \frac{2}{\pi} \sin \frac{t_{k,n} + t_{l,n}}{2} \left(\frac{(2k-1)\pi}{2n} - \frac{(2l-1)\pi}{2n} \right) \\ &\geqslant \frac{k-l}{n} (\sin t_{k,n} + \sin t_{l,n}) \geqslant \frac{k-l}{n} \sin t_{k,n}. \end{aligned}$$

The distance between a_l and a_k is

$$\begin{aligned} a_l - a_k &= x_{l,n} - x_{k,n} - (x_{l,n} - a_l) + (x_{k,n} - a_k) \\ &\geqslant (k-l) \left(\frac{\sin t_{k,n}}{n} + \frac{\sin t_{l,n}}{n} \right) - C_1 \frac{\sin t_{k,n}}{n} - C_1 \frac{\sin t_{l,n}}{n} > 0 \end{aligned}$$

for $k-l=[C_1]+1$. On the other hand

$$
\begin{aligned}
a_l - a_k &\leqslant |a_k - x_{k,n}| + |x_{k,n} - x_{l,n}| + |x_{l,n} - a_l| \\
&\leqslant C_1 \frac{\sin t_{k,n}}{n} + |x_{k,n} - x_{l,n}| + C_1 \frac{\sin t_{l,n}}{n} \\
&\leqslant C_4 \frac{\sin t_{k,n}}{n},
\end{aligned}
$$

where C_4 is another absolute constant. In fact, according to (8),

$$
\begin{aligned}
x_{l,n} - x_{k,n} &= \cos t_{l,n} - \cos t_{k,n} \\
&= 2 \sin \frac{t_{k,n} - t_{l,n}}{2} \sin \frac{t_{k,n} + t_{l,n}}{2} \\
&\leqslant \frac{(k-l)\pi}{n} \sin \frac{t_{k,n} + t_{l,n}}{2},
\end{aligned}
$$

and

$$
\begin{aligned}
\sin \frac{t_{k,n} + t_{l,n}}{2} &= \sin\left(t_{k,n} + \frac{t_{l,n} - t_{k,n}}{2}\right) \\
&= \sin t_{k,n} \cos \frac{t_{l,n} - t_{k,n}}{2} + \cos t_{k,n} \sin \frac{t_{l,n} - t_{k,n}}{2} \\
&\leqslant \sin t_{k,n} + \frac{(k-l)\pi}{n} \\
&\leqslant (1 + (k-l)\pi) \sin t_{k,n} \leqslant (1 + (C_1 + 1)\pi) \sin t_{k,n},
\end{aligned}
$$

since $\sin t_{k,n} \geqslant 1/n$. Similarly

$$
\begin{aligned}
\sin t_{l,n} &= \sin(t_{k,n} + t_{l,n} - t_{k,n}) \\
&= \sin t_{k,n} \cos(t_{l,n} - t_{k,n}) + \cos t_{k,n} \sin(t_{l,n} - t_{k,n}) \\
&\leqslant \sin t_{k,n} + t_{k,n} - t_{l,n} \leqslant \sin t_{k,n} + \frac{(k-l)\pi}{n} \\
&\leqslant (1 + (C_1 + 1)\pi) \sin t_{k,n}.
\end{aligned}
$$

By choosing points $a_k > a_l$, spaced in this way, we can extract a decreasing subsequence, which we again denote $\{a_i\}_1^m$, such that

$$
\begin{gathered}
1 > a_1 > a_2 > \cdots > a_m > -1, \\
0 < a_{i-1} - a_i \leqslant \frac{C_4}{n} \sin t_{i,n} \leqslant \frac{2C_4}{n} \sqrt{1 - |a_i|} \leqslant \frac{C_5}{n} \sqrt{1 - a_i^2}, \qquad (11) \\
1 - |a_m| \leqslant \frac{C_6}{n^2},
\end{gathered}
$$

where C_5 and C_6 are absolute constants. Here the second sequence of inequalities follows from (9) and the fact that $|a_i| < 1$. The last estimate follows from

$$1 - |a_m| = 1 - |x_{m,n}| + |x_{m,n}| - |a_m| \leqslant 1 - x_{m,n}^2 + |x_{m,n} - a_m|$$

$$\leqslant \sin^2 t_{m,n} + \frac{C_1}{n} \sin t_{m,n} \leqslant (1 + C_1) \sin^2 t_{m,n} \leqslant (1 + C_1)(\pi - t_{m,n})^2,$$

since $1/n \leqslant \sin t_{m,n}$, and $\sin t_{m,n} \leqslant \pi - t_{m,n}$. Hence

$$1 - |a_m| \leqslant (1 + C_1)\, \pi^2 \left(1 - \frac{2m-1}{2n}\right)^2 = \frac{1 + C_1}{n^2}\, \pi^2 (2n - 2m + 1)^2$$

$$= \frac{1 + C_1}{n^2}\, \pi^2 (2r + 1)^2 \leqslant \frac{1 + C_1}{n^2}\, \pi^2 (4C_1 + 5)^2.$$

We shall use this sequence in Theorem 2.

THEOREM 2. *Let f be a convex nondecreasing function on $[-1, 1]$ such that*

$$f(-1) = 0, \qquad f(1) = 1.$$

Then there exist points

$$-1 = a_{m+1} < a_m < a_{m-1} < \cdots < a_1 < a_0 = 1,$$

and absolute constants C_5, C_6, and C_7 such that

$$|a_i - a_{i-1}| \leqslant C_5 \frac{\sqrt{1 - a_i^2}}{n}, \qquad i = 1, ..., m,$$

and

$$1 - |a_m| \leqslant \frac{C_6}{n^2}$$

and there exists an interpolating polygon h through the points $(a_i, f(a_i))$, such that

(a) $h(x) \geqslant f(x)$, $x \in [-1, 1]$;

(b) $h(x) = \sum_{i=1}^{m+1} \mu_i g_{a_i}(x)$, *where*

$$\mu_i = (1 - a_i)(a_{i-1} - a_{i+1})\, f[a_{i+1}; a_i; a_{i-1}], \qquad i = 1, 2, ..., m,$$

$$\mu_{m+1} = \frac{2f(a_m)}{1 + a_m};$$

(c) $\|f - h\|_p \leqslant C_7 n^{-2/p}$, $1 \leqslant p < \infty$.

Proof. Let us consider the sequence from (11). Denote by h the interpolating polygon through the points

$$-1=a_{m+1}<a_m<\cdots<a_1<a_0=1,$$

i.e., if $x\in(a_i,a_{i-1})$ then

$$h(x)=f(a_i)\frac{a_{i-1}-x}{a_{i-1}-a_i}+f(a_{i-1})\frac{x-a_i}{a_{i-1}-a_i}$$

and

$$h(x)=\sum_{i=1}^{m+1}\mu_i g_{a_i}(x),$$

where

$$\mu_i=(1-a_i)(a_{i-1}-a_{i+1})f[a_{i+1};a_i;a_{i-1}],\qquad i=1,2,...,m$$

$$\mu_{m+1}=\frac{2f(a_m)}{1+a_m}.$$

Here $\mu_i\geqslant 0$ from the convexity of f and $\sum_{i=1}^{m+1}\mu_i=1$, because

$$1=f(1)=h(1)=\sum_{i=1}^{m+1}\mu_i g_{a_i}(1)=\sum_{i=1}^{m+1}\mu_i.$$

Obviously $h(x)\geqslant f(x)$, $x\in[-1,1]$. Thus

$$\begin{aligned}\|f-h\|_1&=\int_{-1}^{1}(h(x)-f(x))\,dx\\&=\sum_{i=1}^{m+1}\int_{a_i}^{a_{i-1}}(h(x)-f(x))\,dx\leqslant\sum_{i=1}^{m+1}\frac{\mu_i(a_i-a_{i-1})^2}{2(1-a_i)}\\&\leqslant\sum_{i=1}^{m}\frac{\mu_i}{2(1-a_i)}\left(\frac{C_5\sqrt{1-a_i^2}}{n}\right)^2+\frac{\mu_{m+1}(1-|a_m|)^2}{4}\leqslant C_7 n^{-2}.\end{aligned}$$

Here the first inequality follows from convexity in the following way: For $a_i\leqslant x\leqslant a_{i-1}$ we have $h(x)=\sum_{j=i}^{m+1}\mu_j g_{a_j}(x)\geqslant f(x)\geqslant\sum_{j=i+1}^{m+1}\mu_j g_{a_j}(x)$, and thus

$$\begin{aligned}\int_{a_i}^{a_{i-1}}(h(x)-f(x))\,dx&\leqslant\int_{a_i}^{a_{i-1}}\mu_i g_{a_i}(x)\,dx\\&=\frac{\mu_i}{1-a_i}\int_{a_i}^{a_{i-1}}(x-a_i)\,dx=\frac{\mu_i}{1-a_i}\,\frac{(a_i-a_{i-1})^2}{2}.\end{aligned}$$

The estimation of $\|f-h\|_p$ follows trivially as before. Theorem 2 is proved.

Proof of the Main Theorem. Let $f \in K[-1, 1]$. If f is decreasing, the result follows from Theorem 2 by changing x to $-x$. If f is neither increasing nor decreasing it has a minimum $=0$ at a point $x_0 \in (-1, 1)$. Define f_1 by $f_1(x) = f(x)$ for $-1 \leqslant x \leqslant x_0$, $f_1(x) = 0$ for $x_0 \leqslant x \leqslant 1$, and set $f - f_1 = f_2$. For any n there are convex polynomials Q_n^1 and Q_n^2 such that $\|f_i - Q_n^i\|_p \leqslant C_7 n^{-2/p}$, $i = 1, 2$. Then $Q_n = Q_n^1 + Q_n^2$ is again a convex polynomial, and $\|f - Q_n\|_p \leqslant 2C_7 n^{-2/p}$, which proves the theorem.

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References

1. K. G. Ivanov, Approximation of convex functions by means of polynomials and polygons in L-metric, "Proc. Conf. of Functional Spaces and Approximation Theory Gdańsk, 1979," pp. 287–293.
2. M. P. Stojanova, Approximation of a convex function by algebraic polynomials in $L_p[a, b]$ $(1 < p < \infty)$, *Serdica* **11** (1985), 392–397.